Appendix to:
Interethnic conflict and the potential dangers of cross-group ties

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1 Being tipped off about punishment

At any point in time, the number of people in group B who know that a punishment phase has been kicked off depends on the number of people that news of the initial offense has reached from the member of B party to the offense plus the number of people that learned by being punished when randomly paired with a member of A who knew. In other words, the number depends on how widely news spreads from the member of B party to the initial offense and the number that learn from A, which depends on the number in A who learned about the initial offense and the number that learn from B. We can write this recursive relationship precisely.

The number of members of B that $i \in A$ expects will know in $t + l$ about a punishment phase kicked off by a defection in a round between $j \in A$ and $k \in B$ in t when g includes one cross-group tie incident to i and k is:

For $l = 1$:

$$\#BKNOW_{i,1}^{j,k} = \#N_j^r,$$

and for $1 < l \leqslant T$:

$$\#BKNOW_{i,l}^{j,k} = \mathop{\mathbb{E}}_{b_1,\ldots,b_{l-1}} \# \left[N_j^{lr} \cup \left[\bigcup_{m \in b_1} N_m^{(l-1)r} \right] \cup \cdots \cup \left] \bigcup_{m \in b_{l-1}} N_m^r \right] \right]$$

where

$$\begin{aligned} b_1 &\subset B : |b_1| = \#AKNOW_{i,1}^{j,k} \\ &\vdots \\ b_{l-1} &\subset B : |b_{l-1}| = \#AKNOW_{i,l-1}^{j,k}, \end{aligned}$$

and, for $l = 1$,

$$\#AKNOW_{i,l}^{j,k} = \#N_i^r$$

and for $1 < l \leqslant T$:

$$\#AKNOW_{i,l}^{j,k} = \mathop{\mathbb{E}}_{a_1,\ldots,a_{l-1}} \# \left[N_i^{lr} \cup \left[\bigcup_{m \in a_1} N_m^{(l-1)r} \right] \cup \cdots \cup \left] \bigcup_{m \in a_{l-1}} N_m^r \right] \right]$$

where

$$\begin{aligned} a_1 &\subset A : |b_1| = \#BKNOW_{i,1}^{j,k} \\ &\vdots \\ a_{l-1} &\subset A : |b_{l-1}| = \#BKNOW_{i,l-1}^{j,k}. \end{aligned}$$

The number of members of B that $i \in A$ expects will know in $t + l$ about a punishment phase kicked off by a defection in a round between $j \in A$ and $k \in B$ in t when g does *not* contain a cross-group tie incident to i and k is:

For $l = 1$:

$$\#BKNOW_{i,1}^{j,k} = \#\overline{N}_B^r,$$

and for $1 < l \leqslant T$:

$$\#BKNOW_{i,l}^{j,k} = \mathop{\mathbb{E}}_{b_1,\ldots,b_{l-1}} \# \left[\left[\overline{N}_B^{lr} \cup \left[\bigcup_{m \in b_1} N_m^{(l-1)r} \right] \cup \cdots \cup \right] \bigcup_{m \in b_{l-1}} N_m^r \right] \right]$$

where

$$\begin{aligned} b_1 \quad & \subset B : |b_1| = \#AKNOW_{i,1}^{j,k} \\ \vdots \quad & \\ b_{l-1} \quad & \subset B : |b_{l-1}| = \#AKNOW_{i,l-1}^{j,k}, \end{aligned}$$

and, for $l = 1$,

$$\#AKNOW_{i,l}^{j,k} = \#N_i^r$$

and for $1 < l \leqslant T$:

$$\#AKNOW_{i,l}^{j,k} = \mathop{\mathbb{E}}_{a_1,\ldots,a_{l-1}} \# \left[\left[N_i^{lr} \cup \left[\bigcup_{m \in a_1} N_m^{(l-1)r} \right] \cup \cdots \cup \right] \bigcup_{m \in a_{l-1}} N_m^r \right] \right]$$

where

$$\begin{aligned} a_1 \quad & \subset A : |b_1| = \#BKNOW_{i,1}^{j,k} \\ \vdots \quad & \\ a_{l-1} \quad & \subset A : |b_{l-1}| = \#BKNOW_{i,l-1}^{j,k}. \end{aligned}$$

This set of cumbersome recursive relationships reveal a few simpler relationships. First, when $i \in A$ and $k \in B$ are incident to a cross-group tie, i's estimate of the number in B who know changes in only one way: i uses k's precise neighborhood rather than the average neighborhood in B. Second, immediately following the initial cross-group D, i only must consider the actual or average neighborhood from which the news would spread in B and his own neighborhood in A. More than one round after the initial cross-group D, more and more people will learn through punishing encounters. Third, the more densely connected group B is, the more quickly everyone will learn. Fourth, the more densely connected group A, the more quickly everyone will learn.

2 Proof of Proposition 1

Proof. Suppose $T < \frac{diam_{AB}(g)}{r}$. Consider $\arg\max_{i \in J \cup K} \{\min_{j \in J, k \in K}\{\ell(i,j), \ell(i,k)\}\}$. Then there exists a member j in the same group as i such that news of a deviation by i in t will not reach j until $t + (\max_{i \in J \cup K} \{\min_{j \in J, k \in K}\{\ell(i,j), \ell(i,k)\}\})/r$, i.e. $t + (diam_{AB}(g))/r$. But punishment is supposed to end in t + T, which by supposition is before j has heard. Therefore everyone knows the correct end time for punishment only if $T \geqslant \frac{diam_{AB}(g)}{r}$. □

Proof of Proposition 2

Proof. Without loss of generality, consider an individual in group A, $i \in A$. Individual i can deviate from σ^{NWSPIR} in two ways: (1) he can play D when he believes the two groups are in a state of cooperation, and (2) he can play C when he knows the two groups are in a state of punishment. Consider a deviation according to (1). Deviating yields

$$\alpha + \sum_{l=1}^{T} \delta^l \left[0 + \alpha \left[\frac{\#B - \#BKNOW_l^i}{\#B} \right] \right] + \sum_{m=T+1}^{\infty} \delta^m$$

while complying yields

$$1 + \sum_{l=1}^{\infty} \delta^l.$$

Complying is preferable to deviating iff

$$\alpha - 1 \leqslant \sum_{l=1}^{T} \delta^l \left[1 - \alpha \left[\frac{\#B - \#BKNOW_l^i}{\#B}\right]\right].$$

The same holds for $i \in B$:

$$\alpha - 1 \leqslant \sum_{l=1}^{T} \delta^l \left[1 - \alpha \left[\frac{\#A - \#AKNOW_l^i}{\#A}\right]\right].$$

Consider a deviation according to (2) by $i \in A$. If the network were complete, this deviation would be trivially not preferred. Deviating would guarantee a loss of β while complying would earn 0, making complying always better. When networks are incomplete, however, it is possible that if a player expects that very few others know yet, he may prefer to play C hoping to earn 1, and then hope that by not participating in punishment he allowed the number who still don't know in the other group to remain high so he could earn α the next round. This is an unlikely but possible scenario for very sparse networks. Suppose i thinks the cross-group defection that sparked punishment happened d periods ago. Deviating would offer a mix of 1 and $-\beta$, complying would offer a mix of α and 0, with the mix determined by the number in B who know and do not know that the groups are in a state of punishment in $t + d$. For the rest of the punishment phase, the difference between complying and deviating depends on the expected number of out-group members who would have known about the state of punishment but will not due to the deviation. Call $\Delta\#BKNOW_l^{-i}$ the difference between the number in B who would know that the groups are in the state of punishment in $t + l$ if i participates in punishment in t and the number in B who would know if i does not participate in punishment in t. Then complying is preferred to deviating iff

$$(\alpha - 1)\left(\frac{\#B - \#BKNOW_d^i}{\#B}\right) + \beta\left(\frac{\#BKNOW_d^i}{\#B}\right) \geqslant \sum_{l=1}^{T-d} \delta^l \alpha \left[\frac{\Delta\#BKNOW_l^{-i}}{\#B}\right]$$

for all values of d satisfying $1 \leqslant d < T-1$; and likewise

$$(\alpha-1)\left(\frac{\#A-\#AKNOW_d^i}{\#A}\right)+\beta\left(\frac{\#AKNOW_d^i}{\#A}\right) \geqslant \sum_{l=1}^{T-d}\delta^l\alpha\left[\frac{\Delta\#AKNOW_l^{-i}}{\#A}\right]$$

for $i \in B$ and all values of d satisfying $1 \leqslant d < T-1$.

When these conditions are satisfied for all individuals in both groups and all values of d satisfying $1 \leqslant d < T-1$, then any consistent beliefs over the past play of individuals not learned about extend the behavior to sequential equilibirum. □

Proof of Proposition 3

The proof is identical to the proof of Proposition 2 above.

Proof of Proposition 4

The proof follows immediately from the definition of $\#BKNOW_l^{i,j}$ and $\#AKNOW_l^{i,j}$.